1.14 Tensor Calculus I: Tensor Fields

In this section, the concepts from the calculus of vectors are generalised to the calculus of higher-order tensors.

1.14.1 Tensor-valued Functions

Tensor-valued functions of a scalar

The most basic type of calculus is that of tensor-valued functions of a scalar, for example the time-dependent stress at a point, $\mathbf{S} = \mathbf{S}(t)$. If a tensor $\mathbf{T}$ depends on a scalar t, then the derivative is defined in the usual way,

$$\frac{d\mathbf{T}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{T}(t + \Delta t) - \mathbf{T}(t)}{\Delta t},$$

which turns out to be

$$\frac{d\mathbf{T}}{dt} = \frac{dT_{ij}}{dt} \mathbf{e}_i \otimes \mathbf{e}_j \tag{1.14.1}$$

The derivative is also a tensor and the usual rules of differentiation apply,

$$\begin{aligned}
\frac{d}{dt}(\mathbf{T} + \mathbf{B}) &= \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{B}}{dt} \\
\frac{d}{dt}(\alpha(t)\mathbf{T}) &= \alpha \frac{d\mathbf{T}}{dt} + \frac{d\alpha}{dt}\mathbf{T} \\
\frac{d}{dt}(\mathbf{Ta}) &= \mathbf{T}\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{T}}{dt}\mathbf{a} \\
\frac{d}{dt}(\mathbf{TB}) &= \mathbf{T}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{T}}{dt}\mathbf{B} \\
\frac{d}{dt}\left(\mathbf{T}^{\mathrm{T}}\right) &= \left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}}
\end{aligned}$$

For example, consider the time derivative of $\mathbf{QQ}^{\mathrm{T}}$, where $\mathbf{Q}$ is orthogonal. By the product rule, using $\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$,

$$\frac{d}{dt}\left(\mathbf{QQ}^{\mathrm{T}}\right) = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\left(\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}} = \mathbf{0}$$

Thus, using Eqn. 1.10.3e

$$\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} = -\left(\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}} \tag{1.14.2}$$

which shows that $\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$ is a skew-symmetric tensor.

1.14.2 Vector Fields

The gradient of a scalar field and the divergence and curl of vector fields have been seen in §1.6. Other important quantities are the gradient of vectors and higher order tensors and the divergence of higher order tensors. First, the gradient of a vector field is introduced.

The Gradient of a Vector Field

The gradient of a vector field is defined to be the second-order tensor

$$\boxed{\mathrm{grad}\mathbf{a} \equiv \frac{\partial \mathbf{a}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial a_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j} \quad \textbf{Gradient of a Vector Field} \tag{1.14.3}$$

In matrix notation,

$$\mathrm{grad}\mathbf{a} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix} \tag{1.14.4}$$

One then has

$$\begin{aligned} \mathrm{grad}\mathbf{a}\, d\mathbf{x} &= \frac{\partial a_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \left(dx_k \mathbf{e}_k \right) \\ &= \frac{\partial a_i}{\partial x_j} dx_j \mathbf{e}_i \\ &= d\mathbf{a} \\ &= \mathbf{a}(\mathbf{x} + d\mathbf{x}) - \mathbf{a}(d\mathbf{x}) \end{aligned} \tag{1.14.5}$$

which is analogous to Eqn 1.6.10 for the gradient of a scalar field. As with the gradient of a scalar field, if one writes $d\mathbf{x}$ as $|d\mathbf{x}|\mathbf{e}$, where $\mathbf{e}$ is a unit vector, then

$$\mathrm{grad}\mathbf{a}\,\mathbf{e} = \left(\frac{d\mathbf{a}}{dx} \right)_{\text{in } \mathbf{e} \text{ direction}} \tag{1.14.6}$$

Thus the gradient of a vector field **a** is a second-order tensor which transforms a unit vector into a vector describing the gradient of **a** in that direction.

As an example, consider a space curve parameterised by *s*, with unit tangent vector $\boldsymbol{\tau} = d\mathbf{x}/ds$ (see §1.6.2); one has

$$\frac{d\mathbf{a}}{ds} = \frac{\partial \mathbf{a}}{\partial x_j}\frac{dx_j}{ds} = \frac{\partial \mathbf{a}}{\partial x_j}\left(\boldsymbol{\tau}\cdot\mathbf{e}_j\right) = \left(\frac{\partial \mathbf{a}}{\partial x_j}\otimes\mathbf{e}_j\right)\boldsymbol{\tau} = \operatorname{grad}\mathbf{a}\,\boldsymbol{\tau}.$$

Although for a scalar field $\operatorname{grad}\phi$ is equivalent to $\nabla\phi$, note that the gradient defined in 1.14.3 is *not* the same as $\nabla\otimes\mathbf{a}$. In fact,

$$\left(\nabla\otimes\mathbf{a}\right)^{\mathrm{T}} = \operatorname{grad}\mathbf{a} \tag{1.14.7}$$

since

$$\nabla\otimes\mathbf{a} = \mathbf{e}_i\frac{\partial}{\partial x_i}\otimes a_j\mathbf{e}_j = \frac{\partial a_j}{\partial x_i}\mathbf{e}_i\otimes\mathbf{e}_j \tag{1.14.8}$$

These two different definitions of the gradient of a vector, $\partial a_i/\partial x_j\mathbf{e}_i\otimes\mathbf{e}_j$ and $\partial a_j/\partial x_i\mathbf{e}_i\otimes\mathbf{e}_j$, are both commonly used. In what follows, they will be distinguished by labeling the former as $\operatorname{grad}\mathbf{a}$ (which will be called the gradient of **a**) and the latter as $\nabla\otimes\mathbf{a}$.

Note the following:
- in much of the literature, $\nabla\otimes\mathbf{a}$ is written in the contracted form $\nabla\mathbf{a}$, but the more explicit version is used here.
- some authors define the operation of $\nabla\otimes$ on a vector or tensor $(\bullet)$ not as in 1.14.8, but through $\nabla\otimes(\bullet)\equiv\left(\partial(\bullet)/\partial x_i\right)\otimes\mathbf{e}_i$ so that $\nabla\otimes\mathbf{a} = \operatorname{grad}\mathbf{a} = \left(\partial a_i/\partial x_j\right)\mathbf{e}_i\otimes\mathbf{e}_j$.

Example (The Displacement Gradient)

Consider a particle p_0 of a deforming body at position **X** (a vector) and a neighbouring point q_0 at position $d\mathbf{X}$ relative to p_0, Fig. 1.14.1. As the material deforms, these two particles undergo displacements of, respectively, $\mathbf{u}(\mathbf{X})$ and $\mathbf{u}(\mathbf{X}+d\mathbf{X})$. The final positions of the particles are p_f and q_f. Then

$$\begin{aligned} d\mathbf{x} &= d\mathbf{X} + \mathbf{u}(\mathbf{X}+d\mathbf{X}) - \mathbf{u}(\mathbf{X}) \\ &= d\mathbf{X} + d\mathbf{u}(\mathbf{X}) \\ &= d\mathbf{X} + \operatorname{grad}\mathbf{u}\,d\mathbf{X} \end{aligned}$$

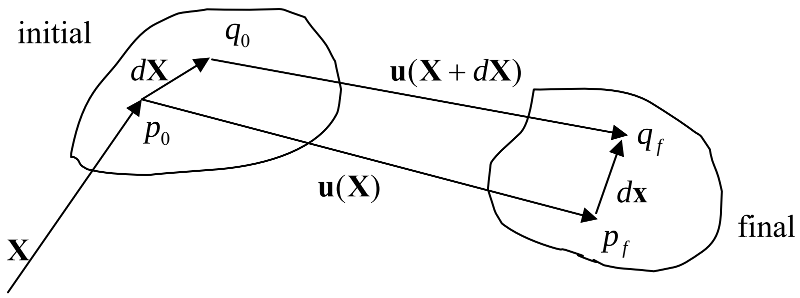

Figure 1.14.1: displacement of material particles

Thus the gradient of the displacement field **u** encompasses the mapping of (infinitesimal) line elements in the undeformed body into line elements in the deformed body. For example, suppose that $u_1 = kX_2^2,\, u_2 = u_3 = 0$. Then

$$\operatorname{grad}\mathbf{u} = \frac{\partial u_i}{\partial X_j} = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2kX_2 \mathbf{e}_1 \otimes \mathbf{e}_2$$

A line element $d\mathbf{X} = dX_i \mathbf{e}_i$ at $\mathbf{X} = X_i \mathbf{e}_i$ maps onto

$$\begin{aligned} d\mathbf{x} &= d\mathbf{X} + \left(2kX_2 \mathbf{e}_1 \otimes \mathbf{e}_2\right)\left(dX_1\mathbf{e}_1 + dX_2\mathbf{e}_2 + dX_3\mathbf{e}_3\right) \\ &= d\mathbf{X} + 2kX_2 dX_2 \mathbf{e}_1 \end{aligned}$$

The deformation of a box is as shown in Fig. 1.14.2. For example, the vector $d\mathbf{X} = d\alpha\mathbf{e}_2$ (defining the left-hand side of the box) maps onto $d\mathbf{x} = 2k\alpha\, d\alpha\mathbf{e}_1 + \alpha\mathbf{e}_2$.

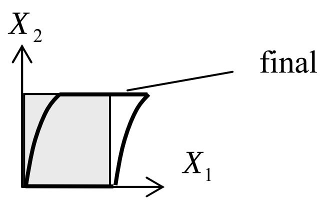

Figure 1.14.2: deformation of a box

Note that the map $d\mathbf{X} \to d\mathbf{x}$ does not specify where in space the line element moves to. It translates too according to $\mathbf{x} = \mathbf{X} + \mathbf{u}$.

■

The Divergence and Curl of a Vector Field

The divergence and curl of vectors have been defined in §1.6.6, §1.6.8. Now that the gradient of a vector has been introduced, one can re-define the divergence of a vector independent of any coordinate system: it is the scalar field given by the trace of the gradient {▲Problem 4},

$$\boxed{\operatorname{div}\mathbf{a} = \operatorname{tr}(\operatorname{grad}\mathbf{a}) = \operatorname{grad}\mathbf{a} : \mathbf{I} = \nabla \cdot \mathbf{a}} \quad \textbf{Divergence of a Vector Field} \quad (1.14.9)$$

Similarly, the curl of **a** can be defined to be the vector field given by twice the axial vector of the antisymmetric part of $\operatorname{grad}\mathbf{a}$.

1.14.3 Tensor Fields

A tensor-valued function of the position vector is called a tensor field, $T_{ij\cdots k}(\mathbf{x})$.

The Gradient of a Tensor Field

The gradient of a second order tensor field **T** is defined in a manner analogous to that of the gradient of a vector, Eqn. 1.14.2. It is the third-order tensor

$$\boxed{\operatorname{grad}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k} \otimes \mathbf{e}_k = \frac{\partial T_{ij}}{\partial x_k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k} \quad \textbf{Gradient of a Tensor Field} \quad (1.14.10)$$

This differs from the quantity

$$\nabla \otimes \mathbf{T} = \mathbf{e}_i \frac{\partial}{\partial x_i} \otimes \left(T_{jk} \mathbf{e}_j \otimes \mathbf{e}_k\right) = \frac{\partial T_{jk}}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \qquad (1.14.11)$$

The Divergence of a Tensor Field

Analogous to the definition 1.14.9, the divergence of a second order tensor **T** is defined to be the vector

$$\boxed{\begin{aligned} \operatorname{div}\mathbf{T} &= \operatorname{grad}\mathbf{T} : \mathbf{I} = \frac{\partial \mathbf{T}}{\partial x_i} \mathbf{e}_i = \frac{\partial (T_{jk} \mathbf{e}_j \otimes \mathbf{e}_k)}{\partial x_i} \mathbf{e}_i \\ &= \frac{\partial T_{ij}}{\partial x_j} \mathbf{e}_i \end{aligned}} \quad \textbf{Divergence of a Tensor} \quad (1.14.12)$$

The divergence of a tensor can also be equivalently defined as that vector field which satisfies the relation

$$(\operatorname{div}\mathbf{T}) \cdot \mathbf{v} = \operatorname{div}\left(\mathbf{T}^{\mathrm{T}} \mathbf{v}\right)$$

for all constant vectors **v**.

One also has

$$\nabla \cdot \mathbf{T} = \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot (T_{jk} \mathbf{e}_j \otimes \mathbf{e}_k) = \frac{\partial T_{ji}}{\partial x_j} \mathbf{e}_i \qquad (1.14.13)$$

so that

$$\text{div}\mathbf{T} = \nabla \cdot \mathbf{T}^{\mathrm{T}} \tag{1.14.14}$$

As with the gradient of a vector, both $\left(\partial T_{ij} / \partial x_j\right)\mathbf{e}_i$ and $\left(\partial T_{ji} / \partial x_j\right)\mathbf{e}_i$ are commonly used as definitions of the divergence of a tensor. They are distinguished here by labelling the former as $\text{div}\mathbf{T}$ (called here the divergence of **T**) and the latter as $\nabla \cdot \mathbf{T}$. Note that the operations $\text{div}\mathbf{T}$ and $\nabla \cdot \mathbf{T}$ are equivalent for the case of **T** symmetric.

Note the following

- some authors define the operation of $\nabla \cdot$ on a vector or tensor $(\bullet)$ not as in (1.14.13), but through $\nabla \cdot (\bullet) \equiv \left(\partial(\bullet) / \partial x_i\right) \cdot \mathbf{e}_i$ so that $\nabla \cdot \mathbf{T} = \text{div}\mathbf{T} = \left(\partial T_{ij} / \partial x_j\right)\mathbf{e}_i$.
- using the convention that the "dot" is omitted in the contraction of tensors, one should write $\nabla\mathbf{T}$ for $\nabla \cdot \mathbf{T}$, but the "dot" is retained here because of the familiarity of this latter notation from vector calculus.
- another operator is the **Hessian**, $\nabla \otimes \nabla = \left(\partial^2 / \partial x_i \partial x_j\right)\mathbf{e}_i \otimes \mathbf{e}_j$.

Identities

Here are some important identities involving the gradient, divergence and curl {▲Problem 5}:

$$\begin{aligned}
\text{grad}(\phi\mathbf{v}) &= \phi\text{grad}\mathbf{v} + \mathbf{v} \otimes \text{grad}\phi \\
\text{grad}(\mathbf{u} \cdot \mathbf{v}) &= (\text{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\text{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u} \\
\text{div}(\mathbf{u} \otimes \mathbf{v}) &= (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{div}\,\mathbf{v})\mathbf{u} \\
\text{curl}(\mathbf{u} \times \mathbf{v}) &= \mathbf{u}\text{div}\mathbf{v} - \mathbf{v}\text{div}\mathbf{u} + (\text{grad}\mathbf{u})\mathbf{v} - (\text{grad}\mathbf{v})\mathbf{u}
\end{aligned} \tag{1.14.15}$$

$$\begin{aligned}
\text{div}(\phi\mathbf{A}) &= \mathbf{A}\text{grad}\phi + \phi\text{div}\mathbf{A} \\
\text{div}(\mathbf{A}\mathbf{v}) &= \mathbf{v} \cdot \text{div}\mathbf{A}^{\mathrm{T}} + \text{tr}(\mathbf{A}\text{grad}\mathbf{v}) \\
\text{div}(\mathbf{A}\mathbf{B}) &= \mathbf{A}\text{div}\mathbf{B} + \text{grad}\mathbf{A} : \mathbf{B} \\
\text{div}(\mathbf{A}(\phi\mathbf{B})) &= \phi\text{div}(\mathbf{A}\mathbf{B}) + \mathbf{A}(\mathbf{B}\text{grad}\phi) \\
\text{grad}(\phi\mathbf{A}) &= \phi\text{grad}\mathbf{A} + \mathbf{A} \otimes \text{grad}\phi
\end{aligned} \tag{1.11.16}$$

Note also the following identities, which involve the Laplacian of both vectors and scalars:

$$\begin{aligned}
\nabla^2(\mathbf{u} \cdot \mathbf{v}) &= \nabla^2\mathbf{u} \cdot \mathbf{v} + 2\text{grad}\mathbf{u} : \text{grad}\mathbf{v} + \mathbf{u} \cdot \nabla^2\mathbf{v} \\
\text{curl}\,\text{curl}\,\mathbf{u} &= \text{grad}(\text{div}\,\mathbf{u}) - \nabla^2\mathbf{u}
\end{aligned} \tag{1.14.17}$$

1.14.4 Cylindrical and Spherical Coordinates

Cylindrical and spherical coordinates were introduced in §1.6.10 and the gradient and Laplacian of a scalar field and the divergence and curl of vector fields were derived in terms of these coordinates. The calculus of higher order tensors can also be cast in terms of these coordinates.

For example, from 1.6.30, the gradient of a vector in cylindrical coordinates is $\mathrm{grad}\mathbf{u} = (\nabla \otimes \mathbf{u})^{\mathrm{T}}$ with

$$\begin{aligned}
\mathrm{grad}\mathbf{u} &= \left[\left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \otimes \left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z\right)\right]^{\mathrm{T}} \\
&= \frac{\partial u_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\mathbf{e}_r \otimes \mathbf{e}_\theta + \frac{\partial u_r}{\partial z}\mathbf{e}_r \otimes \mathbf{e}_z \\
&\quad + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{\partial u_\theta}{\partial z}\mathbf{e}_\theta \otimes \mathbf{e}_z \\
&\quad + \frac{\partial u_z}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\mathbf{e}_z \otimes \mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z \otimes \mathbf{e}_z
\end{aligned} \tag{1.14.18}$$

and from 1.6.30, 1.14.12, the divergence of a tensor in cylindrical coordinates is {▲Problem 6}

$$\begin{aligned}
\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A}^{\mathrm{T}} &= \left(\frac{\partial A_{rr}}{\partial r} + \frac{1}{r}\frac{\partial A_{r\theta}}{\partial \theta} + \frac{\partial A_{rz}}{\partial z} + \frac{A_{rr} - A_{\theta\theta}}{r}\right)\mathbf{e}_r \\
&\quad + \left(\frac{\partial A_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{\theta z}}{\partial z} + \frac{A_{\theta r} + A_{r\theta}}{r}\right)\mathbf{e}_\theta \\
&\quad + \left(\frac{\partial A_{zr}}{\partial r} + \frac{A_{zr}}{r} + \frac{1}{r}\frac{\partial A_{z\theta}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z}\right)\mathbf{e}_z
\end{aligned} \tag{1.14.19}$$

1.14.5 The Divergence Theorem

The divergence theorem 1.7.12 can be extended to the case of higher-order tensors. Consider an arbitrary differentiable tensor field $T_{ij\cdots k}(\mathbf{x},t)$ defined in some finite region of physical space. Let S be a closed surface bounding a volume V in this space, and let the outward normal to S be $\mathbf{n}$. The divergence theorem of Gauss then states that

$$\int_S T_{ij\cdots k} n_k \, dS = \int_V \frac{\partial T_{ij\cdots k}}{\partial x_k} dV \tag{1.14.20}$$

For a second order tensor,

$$\int_S \mathbf{T}\mathbf{n} \, dS = \int_V \mathrm{div}\,\mathbf{T}\, dV, \qquad \int_S T_{ij} n_j \, dS = \int_V \frac{\partial T_{ij}}{\partial x_j} dV \tag{1.14.21}$$

One then has the important identities {▲Problem 7}

$$\begin{aligned}\int_S (\phi\mathbf{T})\mathbf{n}\,dS &= \int_V \operatorname{div}(\phi\mathbf{T})\,dV \\ \int_S \mathbf{u}\otimes\mathbf{n}\,dS &= \int_V \operatorname{grad}\mathbf{u}\,dV \\ \int_S \mathbf{u}\cdot\mathbf{T}\mathbf{n}\,dS &= \int_V \operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{u})\,dV\end{aligned} \qquad (1.14.22)$$

1.14.6 Formal Treatment of Tensor Calculus

Following on from §1.6.12, here a more formal treatment of the tensor calculus of fields is briefly presented.

Vector Gradient

What follows is completely analogous to Eqns. 1.6.46-49.

A **vector field** $\mathbf{v}: E^3 \to V$ is **differentiable** at a point $\mathbf{x}\in E^3$ if there exists a second order tensor $D\mathbf{v}(\mathbf{x})\in E$ such that

$$\mathbf{v}(\mathbf{x}+\mathbf{h}) = \mathbf{v}(\mathbf{x}) + D\mathbf{v}(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|) \quad \text{for all} \quad \mathbf{h}\in E \qquad (1.14.23)$$

In that case, the tensor $D\mathbf{v}(\mathbf{x})$ is called the **derivative** (or **gradient**) of **v** at **x** (and is given the symbol $\nabla\mathbf{v}(\mathbf{x})$).

Setting $\mathbf{h}=\varepsilon\mathbf{w}$ in 1.14.23, where $\mathbf{w}\in E$ is a unit vector, dividing through by ε and taking the limit as $\varepsilon\to 0$, one has the equivalent statement

$$\nabla\mathbf{v}(\mathbf{x})\mathbf{w} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathbf{v}(\mathbf{x}+\varepsilon\mathbf{w}) \quad \text{for all} \quad \mathbf{w}\in E \qquad (1.14.24)$$

Using the chain rule as in §1.6.11, Eqn. 1.14.24 can be expressed in terms of the Cartesian basis $\{\mathbf{e}_i\}$,

$$\nabla\mathbf{v}(\mathbf{x})\,\mathbf{w} = \frac{\partial v_i}{\partial x_k} w_k \mathbf{e}_i = \frac{\partial v_i}{\partial x_j}(\mathbf{e}_i\otimes\mathbf{e}_j) w_k \mathbf{e}_k \qquad (1.14.25)$$

This must be true for all **w** and so, in a Cartesian basis,

$$\nabla\mathbf{v}(\mathbf{x}) = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j \qquad (1.14.26)$$

which is Eqn. 1.14.3.

1.14.7 Problems

1. Consider the vector field $\mathbf{v} = x_1^2\mathbf{e}_1 + x_3^2\mathbf{e}_2 + x_2^2\mathbf{e}_3$. (a) find the matrix representation of the gradient of **v**, (b) find the vector $(\mathrm{grad}\mathbf{v})\mathbf{v}$.
2. If $\mathbf{u} = x_1x_2x_3\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + x_1\mathbf{e}_3$, determine $\nabla^2\mathbf{u}$.
3. Suppose that the displacement field is given by $u_1 = 0, u_2 = 1, u_3 = X_1$. By using $\mathrm{grad}\mathbf{u}$, sketch a few (undeformed) line elements of material and their positions in the deformed configuration.
4. Use the matrix form of $\mathrm{grad}\mathbf{u}$ and $\nabla \otimes \mathbf{u}$ to show that the definitions
   (i) $\mathrm{div}\mathbf{a} = \mathrm{tr}(\mathrm{grad}\mathbf{a})$
   (ii) $\mathrm{curl}\mathbf{a} = 2\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the axial vector of the skew part of $\mathrm{grad}\mathbf{a}$
   agree with the definitions 1.6.17, 1.6.21 given for Cartesian coordinates.
5. Prove the following:
   (i) $\mathrm{grad}(\phi\mathbf{v}) = \phi\mathrm{grad}\mathbf{v} + \mathbf{v} \otimes \mathrm{grad}\phi$
   (ii) $\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) = (\mathrm{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\mathrm{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u}$
   (iii) $\mathrm{div}(\mathbf{u}\otimes\mathbf{v}) = (\mathrm{grad}\,\mathbf{u})\mathbf{v} + (\mathrm{div}\,\mathbf{v})\mathbf{u}$
   (iv) $\mathrm{curl}(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\mathrm{div}\mathbf{v} - \mathbf{v}\mathrm{div}\mathbf{u} + (\mathrm{grad}\mathbf{u})\mathbf{v} - (\mathrm{grad}\mathbf{v})\mathbf{u}$
   (v) $\mathrm{div}(\phi\mathbf{A}) = \mathbf{A}\mathrm{grad}\phi + \phi\mathrm{div}\mathbf{A}$
   (vi) $\mathrm{div}(\mathbf{A}\mathbf{v}) = \mathbf{v}\cdot\mathrm{div}\mathbf{A}^{\mathrm{T}} + \mathrm{tr}(\mathbf{A}\mathrm{grad}\mathbf{v})$
   (vii) $\mathrm{div}(\mathbf{AB}) = \mathbf{A}\mathrm{div}\mathbf{B} + \mathrm{grad}\mathbf{A} : \mathbf{B}$
   (viii) $\mathrm{div}(\mathbf{A}(\phi\mathbf{B})) = \phi\mathrm{div}(\mathbf{AB}) + \mathbf{A}(\mathbf{B}\mathrm{grad}\phi)$
   (ix) $\mathrm{grad}(\phi\mathbf{A}) = \phi\mathrm{grad}\mathbf{A} + \mathbf{A}\otimes\mathrm{grad}\phi$
6. Derive Eqn. 1.14.19, the divergence of a tensor in cylindrical coordinates.
7. Deduce the Divergence Theorem identities in 1.14.22 [Hint: write them in index notation.]